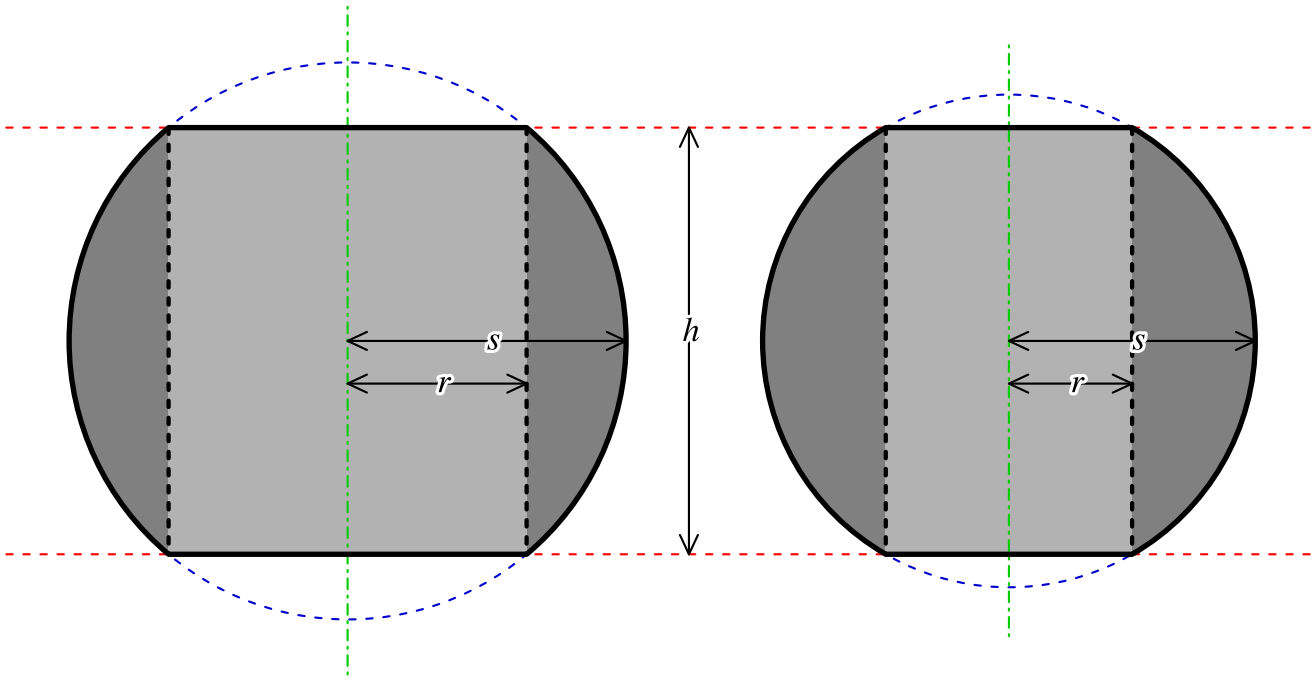


≪Napkin ring problem≫ : a cylindrical hole is bored through the center of a sphere ; what is the volume of the remaining ring ? For rings with matching height h, does the volume vary with sphere radius s ?

Calculating the ring's volume by integrating the surface area of cylindrical shells, with height $2\sqrt{s^2 - x^2}$ and circumference $2\pi x$,

$$\text{ring volume} \;=\; \int_r^s 2\sqrt{s^2 - x^2} \;\cdot\; 2\pi x \, dx$$

substituting $u = x^2$, with $du = 2x \;\; dx$,

$$\begin{aligned}
\text{ring volume} \;&=\; 2\pi \int_{r^2}^{s^2} \sqrt{s^2 - u} \; du \\
&=\; 2\pi \left[\frac{-2}{3} \; \left(s^2 - u\right)^{\frac{3}{2}} \right]_{u=r^2}^{u=s^2} \\
&=\; \frac{4\pi}{3} \left(s^2 - r^2\right)^{\frac{3}{2}} \\
&=\; \frac{4\pi}{3} \left(\frac{h}{2}\right)^3 \\
&=\; \frac{\pi}{6} h^3
\end{aligned}$$

which expresses volume as a function of h irrespective of sphere radius s.